COBEM 99
XV Congresso Brasileiro de Engenharia Mecânica
15th Brazilian Congress of Mechanical Engineering
22 - 26 de Novembro de 1999 / November 22 - 26, 1999 Águas de Lindóia, São Paulo,

# BI-IMPULSIVE ORBITAL TRANSFERS BETWEEN NON-COPLANAR ORBITS WITH MINIMUM TIME FOR A PRESCRIBED FUEL CONSUMPTION

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***Abstract***. *In this work we consider the problem of two-impulse orbital transfers between non-coplanar elliptical orbits with minimum time for a prescribed fuel consumption. We used the equations presented by Eckel and Vinh, add some new equations to consider cases with different geometries, and solved those equations to develop a software for orbital maneuvers. This software can be used in the next missions developed by INPE. The original method developed by Eckel and Vinh was presented without numerical results in that paper. Thus, the modifications considering cases with different geometries, the implementation and the solutions using this method are contributions of this work. The software was tested in real applications with success.*

***Keywords:*** *Astrodynamics, Orbital Transfer, Optimal Control.*

## 1. INTRODUCTION

The majority of the spacecrafts that have been placed in orbit around the Earth utilize the basic concept of orbital transfers. During the launch, the spacecraft is placed in a parking orbit distinct from the final orbit for which the spacecraft was designed. Therefore, to reach the desired final orbit the spacecraft must perform orbital transfers. Besides that, the spacecraft orbit must be corrected periodically because there are perturbations acting on the spacecraft. Both maneuvers are usually calculated with minimum fuel consumption but without a time constraint. This time constraint imposes a new characteristic to the problem that rules out the majority of the transfer methods available in the literature: Hohmann (1925), Hoelker and Silber (1959), Gobetz and Doll (1969), Prado (1989), etc. Therefore, the transfer methods must be adapted to this new constraint: Wang (1963), Lion and Handelsman (1968), Gross and Prussing (1974), Prussing (1969, 1970), Prussing and Chiu (1986), Ivashkin and Skorokhodov (1981), Eckel (1982), Eckel and Vinh (1984) Lawden (1993), Taur et al. (1995) In Brazil, we are going to have important applications with the launch of Remote Sensing Satellites RSS1 and RSS2 that belong to the Complete Brazilian Space Mission; and with the launch of the China – Brazil Earth Resources Satellites CBERS1 and CBERS2.

In this work we consider the problem of two-impulse orbital transfers between non-coplanar elliptical orbits with minimum time for a prescribed fuel consumption. This problem is very important because most of spacecrafts utilize a propulsion system only capable of providing a fixed value of velocity increment, and the velocity increment is direct related to the fuel consumption. We used the equations presented by Eckel and Vinh (1984), add some new equations to consider cases with different geometries, and solved those equations to develop a software for orbital maneuvers. This software can be used in the next missions developed by INPE. The original method developed by Eckel and Vinh (1984) was presented without numerical results in that paper, and it is only valid for a specific geometry of the maneuver. Thus, the modifications considering cases with different geometries, the implementation and the solutions using this method are contributions of this work. The software was tested in real applications with success.

## 2. DEFINITION OF THE PROBLEM

The orbital transfer of a spacecraft from an initial orbit to a desired final orbit consists (Marec 1979) in a change of state (position, velocity and mass) of the spacecraft, from initial conditions $\vec{r}_0$, $\vec{v}_0$ and $m_0$ at time $t_0$ to final conditions $\vec{r}_f$, $\vec{v}_f$ and $m_f$ at time $t_f$ ($t_f \geq t_0$) as shown in Fig. 1.

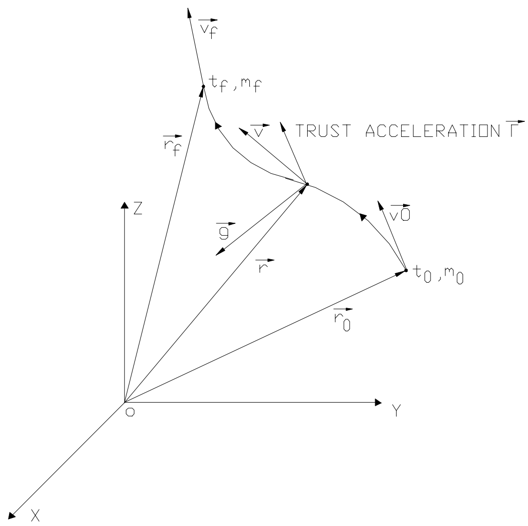


Figure 1 – Orbital Transfer Maneuver, cf. Marec (1979).

The maneuvers can be classified in: maneuvers partially free, when one or more parameter is free (for example, the time spent with the maneuver); or maneuvers completely constrained, when all parameters are constrained. In this case the spacecraft perform an orbital transfer maneuver from a specific point in the initial orbit to another specific point in the final orbit (for example, rendezvous maneuvers). In this work we consider the orbital transfer maneuvers partially free, and that the spacecraft propulsion system is able to apply an impulsive thrust. Therefore, we have the instantaneous variation of the spacecraft velocity.

## 3. PRESENTATION OF THE METHOD

The bases for this method are the equations presented by Eckel and Vinh (1984). These equations furnish the transfer orbit between non-coplanar elliptical orbits with minimum fuel and fixed time transfer, or the transfer orbit with minimum time transfer for a prescribed fuel consumption. But in this work we consider only the problem with minimum time transfer for a prescribed fuel consumption. The problem with minimum fuel and fixed time transfer has already been considered by Rocco (1997) and summarized by Rocco et alli (1999). Therefore, the method was implemented to develop a software for orbital maneuvers. By varying the total velocity increment necessary to the maneuver the software developed furnishes a set of results that are the solution of the problem of bi-impulsive optimal orbital transfer with minimum time for a prescribed fuel consumption.

Given two non-coplanar terminal orbits we desire to obtain a transfer orbit that performs an orbital maneuver from the initial orbit to the final orbit with minimum time and fixed total velocity increment. The orbits are specified by their orbital elements (subscript 1: initial orbit; subscript 2: final orbit; no subscript: transfer orbit):

$a$ = Semi-major axis;
$e$ = Eccentricity;
$p$ = Semi-latus rectum;
$\omega$ = Longitude of the periapsis;
$i$ = Inclination;
$\Omega$ = Longitude of the ascending node;
$M$ = Mean anomaly;
$E$ = Eccentric anomaly.
$\lambda$ = Angle between the planes of the initial and final orbits;
$\beta_1$ = True anomaly of the point $N$ obtained in the plane of the initial orbit;
$\beta_2$ = True anomaly of the point $N$ obtained in the plane of the final orbit;
$I_1$ = Location of the first impulse;
$I_2$ = Location of the second impulse;
$\Delta$ = Transfer angle obtained in the plane of the transfer orbit;
$\gamma_1$ = Plane change angle result of the first impulse;
$\gamma_2$ = Plane change angle result of the second impulse;
$V_1$ = Magnitude of the first impulse;
$V_2$ = Magnitude of the second impulse;
$V$ = Characteristic velocity;
$T$ = Time spent in the maneuver;
$\alpha_1$ = True anomaly of the point $I_1$ obtained in the plane of the initial orbit;
$\alpha_2$ = True anomaly of the point $I_2$ obtained in the plane of the final orbit;
$r_1$ = Distance from point $I_1$;
$r_2$ = Distance from point $I_2$;
$f_1$ = True anomaly of the point $I_1$ obtained in the plane of the transfer orbit;
$f_2$ = True anomaly of the point $I_2$ obtained in the plane of the transfer orbit;
$x_1$ = Radial component of the first impulse;
$x_2$ = Radial component of the second impulse;
$y_1$ = Transverse component of the first impulse in the plane of the initial orbit;
$y_2$ = Transverse component of the second impulse in the plane of the transfer orbit;
$z_1$ = Component of the first impulse orthogonal to the initial orbit;
$z_2$ = Component of the second impulse orthogonal to the transfer orbit;
$h_i$ = Horizontal component of $V_i$.

The geometry of the maneuver is shown in Fig. 2.

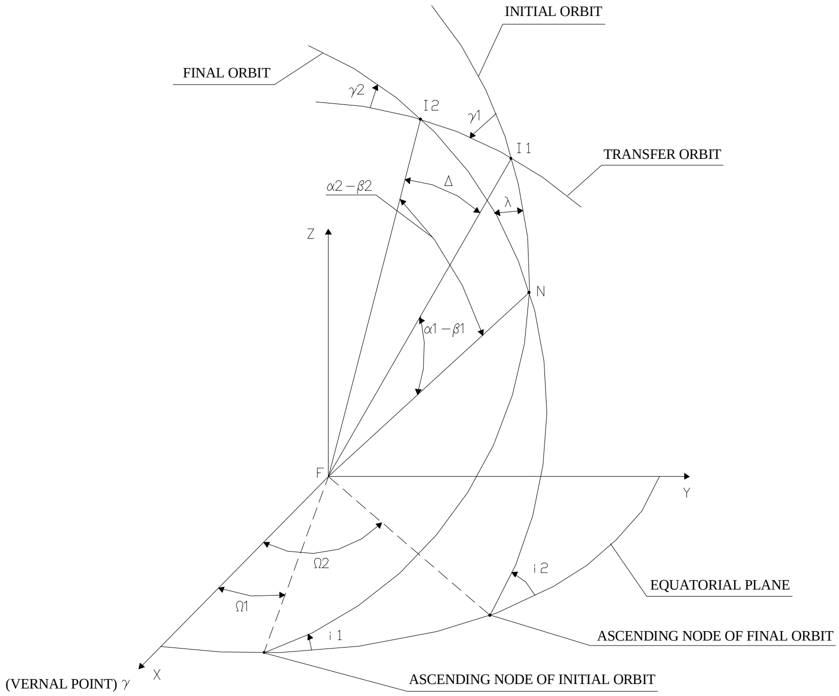


Figure 2 – Maneuver Geometry.

From the geometry of the maneuver we obtain $\beta_1$, $\beta_2$, $\lambda$ and the transfer angle $\Delta$:

$$\beta_1 = \arctan\left[\frac{\sin(\Omega_2 - \Omega_1)\tan(180^\circ - i_2)}{\sin i_1 + \tan(180^\circ - i_2)\cos i_1 \cos(\Omega_2 - \Omega_1)}\right] - \omega_1 \tag{1}$$

$$\beta_2 = \arctan\left[\frac{\sin(\Omega_2 - \Omega_1)\tan i_1}{\sin i_2 + \tan i_1 \cos(180^\circ - i_2)\cos(\Omega_2 - \Omega_1)}\right] - \omega_2 \tag{2}$$

$$\lambda = \arcsin\left[\frac{\sin(\Omega_2 - \Omega_1)\sin i_1}{\sin(\omega_2 + \beta 2)}\right] = \arcsin\left[\frac{\sin(\Omega_2 - \Omega_1)\sin i_2}{\sin(\omega_1 + \beta_1)}\right] \tag{3}$$

$$\cos\Delta = \cos(\beta_1 - \alpha_1)\cos(\alpha_2 - \beta_2) + \sin(\beta_1 - \alpha_1)\sin(\alpha_2 - \beta_2)\cos(180^\circ - \lambda) \tag{4}$$

$$\sin\Delta = \frac{\sin(\alpha_2 - \beta_2)\sin(180^\circ - \lambda)}{\sin B} \tag{5}$$

$$B = \arctan\left[\frac{\sin(180^\circ - \lambda)}{\sin(\beta_1 - \alpha_1)\cot(\alpha_2 - \beta_2) - \cos(\beta_1 - \alpha_1)\cos(180^\circ - \lambda)}\right] \tag{6}$$

Considering that the spacecraft propulsion system is able to apply an impulsive thrust, and that maneuver is bi-impulsive, the total characteristic velocity for the transfer is:

$$V = V_1 + V_2 = F(X) \tag{7}$$

where $X$ is an arbitrary variable for the transfer.

The time of the transfer maneuver is:

$$T = G(X) \tag{8}$$

Therefore, the problem is the minimization of $T$ for a prescribed $V$. If the characteristic velocity is prescribed, being equal to a value $V_0$, we have the constrained relation:

$$V - V_0 = 0 \tag{9}$$

Thus, we have the performance index:

$$J = T + k(V - V_0) \tag{10}$$

From Eckel and Vinh (1984) we know that the solution of the problem depend on three variables: the semi-latus rectum $p$ of the transfer orbit and the true anomaly $\alpha_1$ and $\alpha_2$ that define the position of impulses in the initial and final orbits. Therefore, we have the necessary conditions:

$$\frac{\partial V}{\partial p} + k\frac{\partial T}{\partial p} = 0 \quad ; \quad \frac{\partial V}{\partial \alpha_1} + k\frac{\partial T}{\partial \alpha_1} = 0 \quad ; \quad \frac{\partial V}{\partial \alpha_2} + k\frac{\partial T}{\partial \alpha_2} = 0 \tag{11}$$

By eliminating the Lagrange's multiplier $k$ from equations 11 we have the set of two equations:

$$\frac{\partial V}{\partial p}\frac{\partial T}{\partial \alpha_1} - \frac{\partial V}{\partial \alpha_1}\frac{\partial T}{\partial p} = 0 \quad ; \quad \frac{\partial V}{\partial p}\frac{\partial T}{\partial \alpha_2} - \frac{\partial V}{\partial \alpha_2}\frac{\partial T}{\partial p} = 0 \tag{12}$$

Evaluating the partial derivatives in these equations and doing some simplifications we have the final optimal conditions:

$$\begin{aligned}&(X_1 + YZe\sin f_2)(S_1 q_1 - T_1 e\sin f_1) + S_1 T_1\\ &+ W_1\left(\frac{W_1 - W_2}{\sin \mathit{\Delta}} q_2 - W_1 \tan\frac{\mathit{\Delta}}{2}\right) - \frac{W_1 Z e r_1 e_1 \sin\alpha_1}{q_1 p_1 \sin f_1 \sin\gamma_1} = 0\end{aligned} \tag{13}$$

$$\begin{aligned}&(X_2 + YZe\sin f_1)(S_2 q_2 - T_2 e\sin f_2) + S_2 T_2\\ &+ W_2\left(\frac{W_2 - W_1}{\sin \mathit{\Delta}} q_1 - W_2 \tan\frac{\mathit{\Delta}}{2}\right) + \frac{W_2 Z e r_2 e_2 \sin a_2}{q_2 p_2 \sin f_2 \sin\gamma_2} = 0\end{aligned} \tag{14}$$

which utilize the relations shown in appendix A.

Thus, we have an equation system composed by equations 9, 13 and 14. Solving this equation system by Newton Raphson Method (cf. Press et alli, 1992), we obtain the transfer orbit that performs the maneuver spending a minimum time but with a specific fuel consumption.

## 4. RESULTS

Figures 3 to 8 present some results obtained with the software developed. They come from the maneuver between two terminal non-coplanar elliptical orbits with the following characteristics:

| Initial orbit: | Final orbit: |
|---|---|
| $a_1 = 12030.00000$ km; | $a_2 = 11994.70000$ km; |
| $e_1 = 0.02000$; | $e_2 = 0.01600$; |
| $i_1 = 0.00873$ rad; | $i_2 = 0.00602$ rad; |
| $p_1 = 12025.19000$ km; | $p_2 = 11991.63000$ km; |
| $\omega_1 = 3.17649$ rad; | $\omega_2 = 3.05171$ rad; |
| $\Omega_1 = 0.00000$ rad. | $\Omega_2 = 0.15568$ rad. |

The graphs were obtained through the variation of the total necessary velocity increment for the maneuver from 0.063 to 4.42 km/s. Thus, each point was obtained executing the software to the specific total velocity increment. The points were joined by a line that shows the behavior of that orbital element.

## 5. CONCLUSIONS

In Figs. 3 to 6 we can verify that when the total velocity increment increases the semi-major axis and the eccentricity of the transfer orbit also increase, however, the transfer angle and the time spent in the maneuver decrease. These behaviors occur because when the maneuver is performed with a high value of the velocity increment the transfer orbit approaches a parabolic orbit, so the eccentricity approaches one. Then we have a high value of the semi-major axis and a small value of the transfer angle. In Fig. 7 we have the behavior of the plane change angle. We can verify that when the necessary velocity increment increases the absolute value of the plane change angle also increases. This is expected because changes in inclination, in general, spend more fuel. From this figure we conclude that the sum of the plane change angles almost remain constant because the second impulse undo part of the plane change angle that results of the first impulse. In Fig. 8 we can see that when the maneuver spends more time the velocity increment is smaller than when the maneuver spends less time. This is expected because when the maneuver spends more time the impulse directions approach the movement directions. However, we are studying the non-coplanar case, therefore the impulse directions never will be in the movement directions because we always have a component orthogonal to the orbital plane. In these graphs we can see that it was possible to obtain results when we fixed a small value of the velocity increment, but there is a lower limit, which occur when we reach the solution for time free. Besides that, we can verify that these results are very similar to the results obtained by Rocco (1997) for the case of minimum fuel consumption and fixed time transfer. Thus it is clear that the case of minimum time transfer and fixed fuel consumption is almost the converse of the case of minimum fuel consumption and fixed time transfer. Therefore we conclude that both cases were studied, implemented and tested with success. The simulations shown that the software developed can be used in real applications and it is capable to generate trust results.

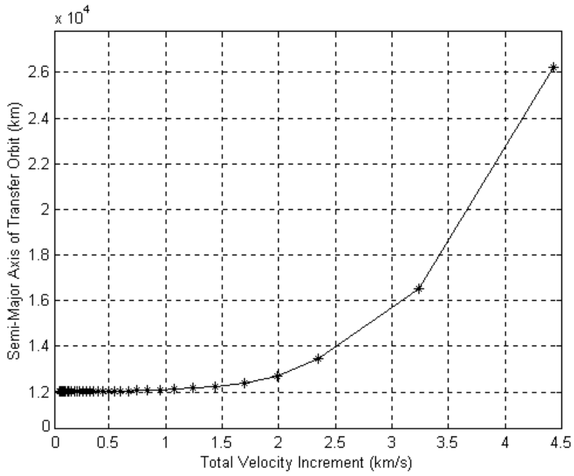


Figure 3 – Semi-Major Axis vs. Total Velocity Increment.

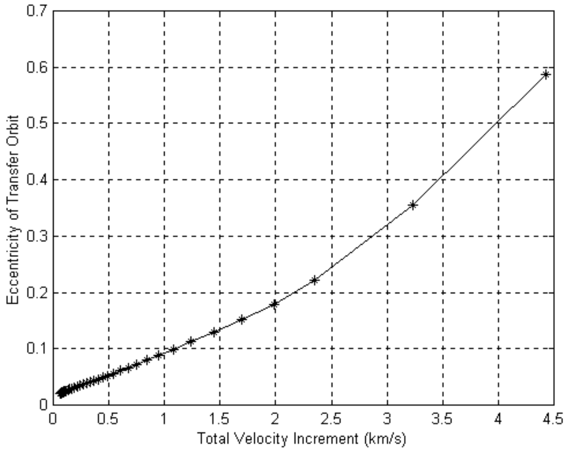


Figure 4 – Eccentricity vs. Total Velocity Increment.

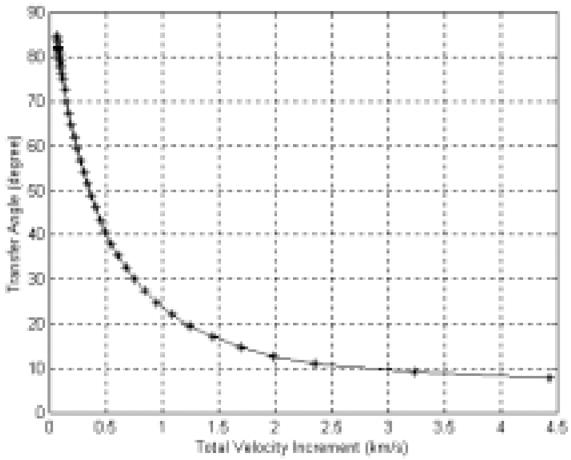


Figure 5 – Transfer Angle vs. Total Velocity Increment.

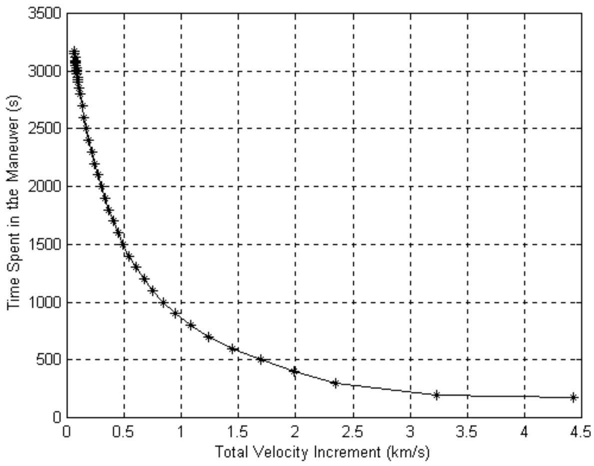


Figure 6 – Time Spent in Maneuver vs. Total Velocity Increment.

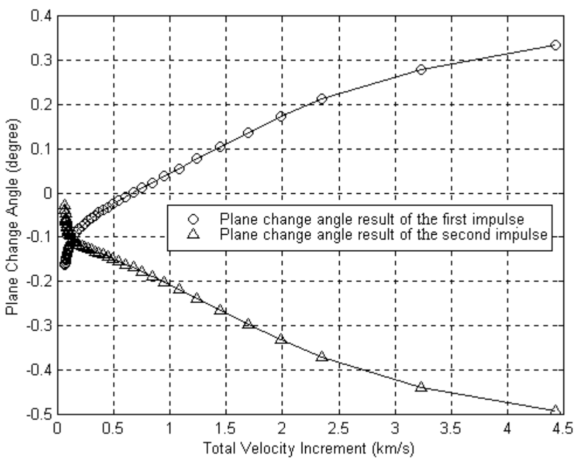


Figure 7 – Plane Change Angle vs. Total Velocity Increment.

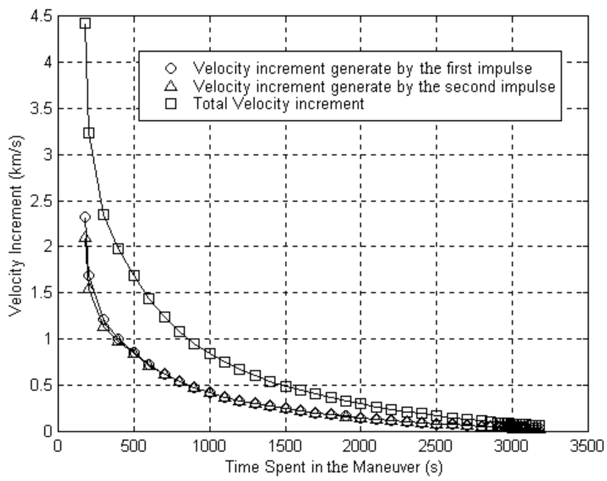


Figure 8 – Velocity Increment vs. Time Spent in Maneuver

## 6. ACKNOWLEDGEMENTS

The authors express their thanks to the National Council for Scientific and Technological Development in Brazil (CNPq) and to the Foundation for Supporting Research in São Paulo State (FAPESP), for supporting this research.

## APPENDIX A

$$r_i = \frac{p_i}{1 + e_i \cos\alpha_i} \tag{A.1}$$

$$\begin{aligned} f_1 &= \arctan\left[\cot\Delta - \frac{r_1(p - r_2)}{r_2(p - r_1)\sin\Delta}\right] \\ f_2 &= \arctan\left[\frac{r_2(p - r_1)}{r_1(p - r_2)\sin\Delta} - \operatorname{cotg}\Delta\right] \end{aligned} \tag{A.2}$$

$$p = \frac{r_1\, r_2\left(\cos f_1 - \cos f_2\right)}{r_1 \cos f_1 - r_2 \cos f_2} \tag{A.3}$$

$$e = \frac{r_2 - r_1}{r_1 \cos f_1 - r_2 \cos f_2} \tag{A.4}$$

$$a = \frac{p}{1 - e^2} \tag{A.5}$$

$$\begin{aligned} \gamma_1 &= \arcsin\left[-\frac{\sin(\beta_2 - \alpha_2)}{\sin\Delta}\sin\phi\right] \\ \gamma_2 &= \arcsin\left[-\frac{\sin(\beta_1 - \alpha_1)}{\sin\Delta}\sin\phi\right] \end{aligned} \tag{A.6}$$

$$\begin{aligned} x_1 &= \sqrt{\mu}\left(\frac{e}{\sqrt{p}}\sin f_1 - \frac{e_1}{\sqrt{p_1}}\sin\alpha_1\right) \\ x_2 &= \sqrt{\mu}\left(\frac{e_2}{\sqrt{p_2}}\sin\alpha_2 - \frac{e}{\sqrt{p}}\sin f_2\right) \end{aligned} \tag{A.7}$$

$$\begin{aligned} y_1 &= \frac{\sqrt{\mu}}{r_1}\left(\sqrt{p} - \sqrt{p_1}\cos\gamma_1\right) \\ y_2 &= \frac{\sqrt{\mu}}{r_2}\left(\sqrt{p_2}\cos\gamma_2 - \sqrt{p}\right) \end{aligned} \tag{A.8}$$

$$z_i = \frac{\sqrt{\mu p_i}}{r_i} \sin \gamma_i \tag{A.9}$$

$$h_i = \left( y_i^{\,2} + z_i^{\,2} \right)^{1/2} \tag{A.10}$$

$$V_i = \left( x_i^{\,2} + h_i^{\,2} \right)^{1/2} \tag{A.11}$$

$$S_i = \frac{x_i}{V_i} \tag{A.13}$$

$$T_i = \frac{y_i}{V_i} \tag{A.14}$$

$$W_i = \frac{z_i}{V_i} \tag{A.15}$$

$$q_i = \frac{p}{r_i} \tag{A.16}$$

$$E_i = \arccos\left( \frac{e + \cos f_i}{1 + e \cos f_i} \right) \tag{A.17}$$

$$\sin E_i = \frac{\sqrt{1 - e^2} \sin f_i}{1 + e \cos f_i} \tag{A.18}$$

$$M_i = E_i - e \sin E_i \tag{A.19}$$

$$T = \sqrt{\frac{a^3}{\mu}} \left( M_2 - M_1 + 2\pi N \right) \tag{A.20}$$

$$\begin{aligned} X_1 &= \frac{S_1 \cos \Delta - S_2}{\operatorname{sen} \Delta} + T_1 \\ X_2 &= \frac{S_1 - S_2 \cos \Delta}{\operatorname{sen} \Delta} + T_2 \end{aligned} \tag{A.21}$$

$$Y = \frac{1}{\left(1 - e^2\right) \operatorname{sen} \Delta} \left[ 3e^2 T \sqrt{\frac{\mu}{p^3}} - 2e \left( \frac{1}{q_2 \operatorname{sen} f_2} - \frac{1}{q_1 \operatorname{sen} f_1} \right) + \operatorname{cotg} f_2 - \operatorname{cotg} f_1 \right] \tag{A.22}$$

$$Z = \frac{q_2\, X_2 - q_1\, X_1 + \left( S_1 + S_2 \right) \operatorname{tg} \Delta/2}{\operatorname{cotg} f_1 - \operatorname{cotg} f_2 + Y \left[ \left( 1 + e^2 \right) \operatorname{sen} \Delta + 2e \left( \operatorname{sen} f_2 - \operatorname{sen} f_1 \right) \right]} \tag{A.23}$$

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